Factoring Polynomials: Equating Coefficients

Equating polynomials is a more efficient way to factor polynomials compared to long division; moreover, the pitfalls of synthetic division could be avoided.

Suppose an m^{th} degree polynomial, $f(x)$, is a factor of an n^{th} degree polynomial, $P(x)$; then $P(x)$ can be written as a product of $f(x)$ and an $(n-m)^{\text{th}}$ degree polynomial, that is,

$$P(x) = f(x) \cdot ((n-m)^{\text{th}} \text{ degree polynomial})$$

The coefficients of the $(n-m)^{\text{th}}$ degree polynomial can be determined by comparing corresponding coefficients on the left and on the right side of the equation above.

For example, $x+1$ is a factor of the polynomial $p(x) = x^3 + x^2 - 4x - 4$; as $p(x)$ is a cubic, it can be written as the product of $x+1$ and a quadratic, that is,

$$\begin{aligned} p(x) &= (ax^2 + bx + c)(x+1) \\ x^3 + x^2 - 4x - 4 &= (ax^2 + bx + c)(x+1) \quad - - - - - - - - -(1) \end{aligned}$$

From inspection, the coefficient of x^3 on the left side is 1 and the coefficient of x^3 on the right side is a; hence $a = 1$.

Equating the constant terms: $c = -4$

Equation (1) can now be written as

$$x^3 + x^2 - 4x - 4 = (x^2 + bx - 4)(x+1)$$

We have equated the coefficients of x^3 and the constant terms; we can now equate the coefficients of x^2 or x to determine the value of b.

Equate the coefficients of x^2: $b + 1 = 1 \quad \Longrightarrow \quad b = 0$

Therefore,

$$\begin{aligned} x^3 + x^2 - 4x - 4 &= (x^2 - 4)(x+1) \\ &= (x+2)(x-2)(x+1) \end{aligned}$$

The method of equating coefficients is particularly useful when one of the factors of $p(x)$ is an irreducible quadratic.

For example, $x^2 + x + 1$ is a factor of $p(x) = 2x^4 + 3x^3 + 2x^2 - 1$; as $p(x)$ is a quartic, it is the product of $x^2 + x + 1$ and another quadratic, that is,

$$2x^4 + 3x^3 + 2x^2 - 1 = (ax^2 + bx + c)(x^2 + x + 1) \quad - - - (2)$$

Equating coefficients of x^4: $a = 2$

Equating constants: $c = -1$

Equation (2) can now be written as

$$2x^4 + 3x^3 + 2x^2 - 1 = (2x^2 + bx - 1)(x^2 + x + 1)$$

In order to determine the value of b, we can either equate the coefficients of x^3 or the coefficients of x^2.

Equating the coefficients of x^3: $2 + b = 3 \quad \Longrightarrow \quad b = 1$

Therefore,

$$\begin{aligned} 2x^4 + 3x^3 + 2x^2 - 1 &= (2x^2 + x - 1)(x^2 + x + 1) \\ &= (2x - 1)(x + 1)(x^2 + x + 1) \end{aligned}$$

Example 1

Factor $p(x) = 3x^3 - 8x^2 - x + 10$ fully.

Solution

We need to consider the factors of 10 and 3 to determine the first zero of $p(x)$.
Factors of 10: $\pm1, \pm2, \pm5, \pm10$
Factors of 3: $\pm1, \pm3$
Possible zeros are: $\pm1, \pm2, \pm5, \pm10, \pm1/3, \pm2/3, \pm5/3, \pm10/3$

- $p(1) = 3(1)^3 - 8(1)^2 - 1 + 10 = 4 \neq 0$
- $p(-1) = 3(-1)^3 - 8(-1)^2 - (-1) + 10 = 0$

As $x + 1$ is a factor of $p(x)$, it can be written as the product of $x + 1$ and a quadratic,

$$3x^3 - 8x^2 - x + 10 = (ax^2 + bx + c)(x + 1)$$

Equating coefficients of x^3: $a = 3$

Equating constants: $c = 10$

It follows that

$$3x^3 - 8x^2 - x + 10 = (3x^2 + bx + 10)(x + 1)$$

Equating coefficients of x: $b + 10 = -1 \quad \Longrightarrow \quad b = -11$

Hence,

$$\begin{aligned} 3x^3 - 8x^2 - x + 10 &= (3x^2 + bx + 10)(x + 1) \\ &= (3x - 5)(x - 2)(x + 1) \end{aligned}$$

Example 2

Solve $x^4 + x^3 - 2x^2 - 6x - 4 = 0$.

Solution

Possible roots are $\pm 1,\ \pm 2,\ \pm 4$

Let $p(x) = x^4 + x^3 - 2x^2 - 6x - 4$

- $p(1) = (1)^4 + (1)^3 - 2(1)^2 - 6(1) - 4 = -10 \neq 0$
- $p(-1) = (-1)^4 + (-1)^3 - 2(-1)^2 - 6(-1) - 4 = 0$

$x + 1$ is a factor of $p(x)$ and $p(x)$ can be written as a product of $x + 1$ and a cubic polynomial, that is,

$$x^4 + x^3 - 2x^2 - 6x - 4 = (ax^3 + bx^2 + cx + d)(x + 1)$$

Equating coefficients of x^4: $a = 1$

Equating constants: $d = -4$

Thus, $p(x)$ can be written as

$$p(x) = x^4 + x^3 - 2x^2 - 6x - 4 = (x^3 + bx^2 + cx - 4)(x + 1)$$

Equating coefficients of x^3: $b + 1 = 1 \quad \Longrightarrow \quad b = 0$

Equating coefficients of x^2: $b + c = -2 \quad \Longrightarrow \quad c = -2$

Now,

$$p(x) = x^4 + x^3 - 2x^2 - 6x - 4 = (x^3 - 2x - 4)(x + 1)$$

The cubic $x^3 - 2x - 4$ should be factored into linear factors, if possible.

Let $f(x) = x^3 - 2x - 4$.

Possible zeroes of $f(x)$ are $\pm 1, \pm 2, \pm 4$.

$f(2) = 2^3 - 2(2) - 4 = 8 - 4 - 4 = 0$, so $x - 2$ is a factor of $f(x)$.

$f(x) = x^3 - 2x - 4$ is a product of $x - 2$ and a quadratic, that is,

$$f(x) = x^3 - 2x - 4 = (ax^2 + bx + c)(x - 2)$$

Equating coefficients of x^3 gives $a = 1$.

Equating constants gives $-2c = -4$, that is $c = 2$.

Equating coefficients of x: $c - 2b = -2 \quad \Longrightarrow \quad b = 2$

As the quadratic, $x^2 + 2x + 2$ is irreducible, the fully factored form of $f(x)$ is

$$f(x) = x^3 - 2x - 4 = (x^2 + 2x + 2)(x - 2)$$

The original quartic in fully factored form is thus

$$x^4 + x^3 - 2x^2 - 6x - 4 = (x + 1)(x - 2)(x^2 + 2x + 2)$$

The solution to $x^4 + x^3 - 2x^2 - 6x - 4 = 0$ is equivalent to the solution to $(x+1)(x-2)(x^2+2x+2) = 0$, that is

$$x = -1, 2$$